KÄHLER-RICCI FLOW AND THE MINIMAL MODEL PROGRAM FOR PROJECTIVE VARIETIES

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Abstract. In this note we propose to show that the Kähler-Ricci flow fits naturally within the context of the Minimal Model Program for projective varieties. In particular we show that the flow detects, in finite time, the contraction theorem of any extremal ray and we analyze the singularities of the metric in the case of divisorial contractions for varieties of general type. In case one has a smooth minimal model of general type (i.e., the canonical bundle is nef and big), we show infinite time existence and analyze the singularities.

1. Introduction

One of the most important problems in Algebraic Geometry is the quest for a Minimal Model (i.e., a variety which is birationally equivalent to the given one whose canonical bundle is *nef*), dubbed "the Minimal Model Program". This entails the application of a complicated algorithm, which has been proved to work in dimension 3 by the collaborative effort of Mori, Kawamata, Kollár, Shokurov et al. (see [Kawamata-Matsuda-Matsuki] for a survey) and very recently in dimension 4 [Hacon-M^{c}Kernan]. In complex dimension 2 the theory is much simpler and is due to the Italian school of algebraic geometry (like Enriques, Castelnuovo and Severi) and put in a modern and more precise framework by Zariski and Kodaira. In case the variety is of *general type* (i.e., varieties for which the space of holomorphic sections of K_M^k grows like k^n, where $n = \dim M$), then out of a minimal model one can produce the so called *canonical model*, i.e., a birational model whose canonical bundle is ample, or, in other words, with negative first Chern class c_1.

On the other hand, around the 1980's, building on the foundational work of Hamilton in the Riemannian case, H. D. Cao studied the Kähler-Ricci flow for canonical metrics on manifolds with definite first Chern class. c_1, reproving (see [Cao]) in particular Calabi's conjecture and the existence of Kähler-Einstein metrics in case $c_1 < 0$ (the solution of the conjecture is originally due to S.T. Yau, cf. [Yau]). Cast in an algebro-geometric light, this last result says that any smooth projective variety with ample canonical bundle admits a Kähler-Einstein metric (see [Demailly-Kollár] if M is a projective variety with orbifold singularities).

In this note we propose to draw a connection between the two theories for projective varieties of general type, and in fact prove that in complex dimension two the Kähler-Ricci flow produces the *canonical model*, generalizing Cao's result.

In order to state the two main theorems we prove, we need to fix some notation. Let M be a projective variety and K_M its *canonical* line bundle. If K_M is not *nef*, there exists a (complex) curve C in M on which $K_M \cdot C < 0$. By the rationality theorem (e.g. see [Kawamata-Matsuda-Matsuki]), there exists a nef line bundle L such that $A = L - rK_M$ is ample, for some rational number $r > 0$. Therefore (by the base-point-free theorem) $L = A + rK_M$ is *semiample*, that is to say some power L^n of L defines a holomorphic map $c : M \to M'$. Moreover the morphism c contracts only the curves that are homologically equivalent to C.

In this context we prove:

Theorem 1.1. *Let M, M' be as above, and let g_0 be a metric in the ample class A. Then the Kähler-Ricci flow:*

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -\operatorname{Ric}_{i\bar{j}} - g_{i\bar{j}}$$

flows the metric in the class $c_1(A + a(t)(K_M - A))$ for $a(t) = 1 - e^{-t}$ and it develops a singularity at (the finite time) $T = \log(r+1)$.

Furthermore, if M is of general type, then the singular locus S of $g(T)$ is contained in a proper subvariety of M. If, in addition, M' is smooth (in particular c is a divisorial contraction), then $g(T)$ induces a smooth metric on M'.

In case M is a smooth minimal model (i.e. K_M is nef), we prove:

Theorem 1.2. *Let M be a smooth projective manifold with K_M big and nef, then the normalized Kähler-Ricci flow as above exists for all time. Moreover, if the canonical model M' of M admits only orbifold singularities, then the limit for $t \to +\infty$ of $g(t)$ is locally equal to $g_0 + dd^c v$ where v is a bounded function which is smooth away from an analytic subvariety.*

In particular if M is a projective surface of general type, then all the hypotesis of the previous two theorems are satisfied, and therefore, after a finite number of steps, the Kähler-Ricci flow converges to a Kähler-Einstein metric on the canonical model of M.

A more general case will be studied in a forthcoming paper.

Let us remark that Theorem 1.2 was a conjecture of Tian and that it has been proved independently (with weaker assumptions) by G. Tian and Z. Zhang (cf. [Tian-Zhang]).

The techniques involved are mainly the reduction of the Kähler-Ricci flow at hand to a scalar parabolic PDE similar to Cao's ([Cao]), and the coarse study of its singularities, in the context of orbifolds.

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Notations and Conventions. On a n-dimensional complex manifold M, with local holomorphic coordinates $z_1, \cdots, z_n$, we will write $d = \partial + \bar{\partial}$ and $d^c = \frac{\sqrt{-1}}{4\pi}(\partial - \bar{\partial})$, so that $\mathrm{dd^c} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$. Given a Kähler metric $g = \frac{\sqrt{-1}}{2\pi}\sum g_{i\bar{j}} dz_i \wedge dz_{\bar{j}}$, we will denote by Ric_g its Ricci curvature, locally given by $\operatorname{Ric}_g = -\mathrm{dd^c}\log\det(g_{ij})$. We will often use matricial notation for $(1,1)$ form, e.g. g^{-1} locally denotes the inverse matrix $g^{i\bar{j}}$ of g. As usual, given two quadratic forms on a given vector space, q_1 and q_2, with q_1 non-singular, we set $tr_{q_1} q_2 = tr(q_1^{-1} q_2)$.

2. The scalar equation

Let (M, g_0) be a projective manifold with a Kähler metric g_0 belonging to the ample class A. The first thing to observe is that if the Kähler metric $g(t)$ satisfies the Kähler-Ricci flow,

$$\begin{cases} g'(t) = -\operatorname{Ric}_{g(t)} - g(t) \\ g(0) = g_0 \end{cases} \tag{1}$$

then, $g(t)$ is in the class of $c_1(A(t))$ where $A(0) = A$ and $A(t)$ is an ample class on M.

By abuse of notation, and since $-\operatorname{Ric}_g \in c_1(K_M)$ for any Kähler metric g, we have

$$\begin{cases} \partial_t A(t) = K_M - A(t) \\ A(0) = A \end{cases}$$

or, therefore, by solving the ODE, we have the following

Proposition 2.1. *If $g(t)$ is a solution for the Kähler-Ricci flow (1), then $g(t)$ belongs to the class*

$$A(t) = A + a(t)(K_M - A)$$

with $a(t) = 1 - e^{-t}$.

With this at hand, let us proceed to reduce our equation to a scalar one.

Let $\eta_0 = -g_0 - \mathrm{Ric}_{g_0}$. In particular, η_0 is an element in the class $K_M - A$. Thus, given any element η in the same class (we will allow ourselves the freedom of choosing η later on, depending on the situation), there exists a smooth function $f = f_\eta$ on M, such that:

$$\eta_0 = \eta + \mathrm{dd^c}\, f.$$

Moreover, if we consider the $(1,1)$-form

$$g_0(t) = g_0 + a(t)\eta, \tag{2}$$

then $g_0(t)$ belongs to the class $A(t)$, and therefore there exists a smooth function u (also depending on η), such that

$$g(t) = g_0(t) + \mathrm{dd^c}\, u.$$

Thus differentiating, and using the fact that $\mathrm{Ric}_{g(t)} = -\,\mathrm{dd^c} \log\big(\det g(t)\big)$, equation (1) becomes

$$a'(t)\,\eta + \mathrm{dd^c}(\partial_t u) = \mathrm{dd^c} \log\, \det\big(\frac{g_0(t) + \mathrm{dd^c}\, u}{g_0}\big) + \eta_0 - a(t)\eta - \mathrm{dd^c}\, u$$

and therefore:

$$\mathrm{dd^c}\,(\partial_t u)) = \mathrm{dd^c} \log\, \det\left(\frac{g_0 + a(t)\eta + \mathrm{dd^c}\, u}{g_0}\right) - \mathrm{dd^c}\, u + \mathrm{dd^c}\, f.$$

Thus, by the $\partial\overline{\partial}$-lemma, there exists a smooth function $\phi(t)$ (depending only on t) such that:

$$\partial_t u = \log\, \det\left(\frac{g_0 + a(t)\eta + \mathrm{dd^c}\, u}{g_0}\right) - u + f + \phi.$$

Moreover ϕ satisfies:

$$\int_M \exp\left(\partial_t u + u - f\right)\, dV_0 = \mathrm{Vol}(M) e^{\phi(t)}$$

where dV_0 is the volume form of the metric g_0, and $\mathrm{Vol}(M)$ is the volume of M with respect to the metric $g(t)$.

Renormalizing u so that $\phi = 0$, we have the following:

Lemma 2.2. *The Kähler-Ricci flow as in theorem 1.1 is equivalent to the scalar equation:*

$$\begin{cases} \partial_t u = \log\, \det\left(\dfrac{g_0 + a(t)\eta + \mathrm{dd^c}\, u}{g_0}\right) - u + f \\ u(x,0) = 0 \end{cases} \tag{3}$$

Notice that the solution u to eq. (3) is dependent on the choice of η, but g is not. In fact we have:

Lemma 2.3. *There exists a one-to-one correspondence between the solutions u of (3) with pair (η, f) and the solutions u' with pair $(\eta', f') = (\eta', f') = (\eta - dd^c h, f + h)$, given by $u' = u + a(t)h$. In particular h does not depend on t.*

3. Maximal existence time Case I: K_M not nef

Given a Kähler manifold (M, g_0), we first investigate the behavior of the solutions to equation (3), in the case M is a projective variety with canonical bundle not nef.

The equation is parabolic and general theory implies short time existence (e.g. see [Taylor]). Moreover we have:

Proposition 3.1. *Suppose u is a solution to eq. (3) in $M \times (0, t_0)$, for some time $t_0 > 0$. Then there exists a uniform constant $C > 0$ such that:*

$$|u| < C.$$

In particular, if the flow exists for any $t < t_0$, then for any sequence $\{t_i\} \subset [0, t_0)$, we have that $\lim_{t_i \to T} u(x, t_i)$ is continuous (up to taking a sub-sequence).

In order to prove the proposition, we have to choose a suitable η. Although u depends on η, from lemma 2.3 it follows that the fact that u is bounded does not depend on that particular choice.

As stated in the introduction, by the rationality theorem and the base point free theorem, since K_M is not nef, there exists a rational number r such that $L = A + rK_M$ is semi-ample and defines a contraction $c : M \to M'$ of an extremal ray. In fact,

$$r = \max\{s \in \mathbb{Q} | A + sK_M \text{ is nef}\}. \tag{4}$$

In particular, by proposition 2.1, it follows that there can be a solution for (1) at most up to time

$$T = \log(r + 1).$$

In fact, the class $A(T) = (r + 1)^{-1}L$ is not ample and therefore it cannot contain any metric.

On the other hand, L is semi-ample, i.e. it is the pull-back of some ample class A' on M' with respect to the contraction map c. Therefore there exists a non-negative $(1, 1)$ form η_L in $c_1(L)$.

By prop. 2.1, we can write

$$A(t) = \frac{1}{r}(a(t)L + b(t)A)$$

where $a(t) = 1 - e^{-t}$ and $b(t) = (r + 1)e^{-t} - 1$. in particular we can choose η as

$$\eta = \frac{1}{r}(\eta_L - (r + 1)g_0), \tag{5}$$

so that the $(1, 1)$ form $g_0(t)$, defined in (2), is given by

$$g_0(t) = g_0 + a(t)\eta = \frac{1}{r}(a(t)\eta_L + b(t)g_0),$$

and therefore it is a metric for any $t < T$.

We need the following

Lemma 3.2. *There exists a bounded super-solution u^+ (resp. a bounded sub-solution u^-) for (3), depending only on the time t, defined for any $t \in [0, T)$, i.e.*

$$\partial_t u^+ + u^+ \geq \log(\det(g_0(t)/g_0)) + f$$

$$\left(\quad \textit{resp.} \quad \partial_t u^- + u^- \leq \log(\det(g_0(t)/g_0)) + f \quad \right).$$

Proof. Since M is compact, and since f is a smooth function on M and $a(t)$ and $b(t)$ are bounded functions on $[0,T)$, there exists a positive constant K such that $\log(\det(g_0(t)/g_0)) + f < K$.

Therefore, in order to define a super-solution for (3), it is enough to choose u^+ as a solution of $\partial_t u^+ + u^+ = K$, with $u^+(0) = 0$, i.e.

$$u^+ = (1 - e^{-t})K.$$

On the other hand, $a(t), b(t)$ are non-negative in the interval $[0,T)$ and since η_L is semi-positive, it follows

$$\det(g_0(t)/g_0) = \det\left(\frac{1}{r}\cdot\frac{a(t)\eta_L + b(t)g_0}{g_0}\right) \geq \left(\frac{b(t)}{r}\right)^n.$$

It can be easily checked that $\displaystyle\int_0^T \log\ b(t)\ dt > -\infty$ and therefore, given K such that $f > K$, we can define u^- as the solution for

$$\begin{cases} \partial_t u^- + u^- = \log\left(b(t)/r\right)^n + K \\ u^-(0) = 0 \end{cases}$$

Thus u^- is a bounded sub-solution for (3). □

Proof of Proposition 3.1.

Let u^- and u^+ be as in lemma 3.2. By the comparison principle, we will show that for any solution $u(t)$ of (3), we have

$$u^-(t) \leq u(t) \leq u^+(t).$$

Since u^- and u^+ are bounded function, the proposition will follow.

Let $w = u - u^-$. Then, since u^+ is a super-solution for (3) that depends only on the time t, we have that for any $t < T$,

$$\partial_t w + w \leq \log(\det(g_0(t) + \mathrm{dd^c}\, w)/g_0(t)).$$

Therefore if $\overline{w}(t) = \max_M(w(\cdot,t))$, then $\partial_t\overline{w} + \overline{w} \leq 0$. Since $w(\cdot,0) = 0$, it follows $w \leq 0$, i.e. $u \leq u^+$. Similarly, it follows $u \geq u^-$. □

Remark 3.3. Let us define

$$F(x,t,r,p,X) = e^{p+r-f(x)} - \det((g_0(t) + X)/g_0).$$

Then the scalar equation (3) for the Kähler-Ricci flow is equivalent to the equation

$$\begin{cases} F(x,t,u(x,t),\partial_t u(x,t), \mathrm{dd^c}\, u(x,t)) = 0 \\ u(x,0) = 0 \end{cases} \tag{6}$$

It is easy to check that the operator F is *proper*, (according to the definition (0.2) and (0.3) in [Crandall-Ishii-Lions]).

Therefore by Perron's method (e.g., see theorem 4.1 in [Crandall-Ishii-Lions]), the existence of a super-solution and a sub-solution for (6), guaranteed by lemma 3.2, implies the existence of a weak solution (in the sense of viscosity solutions) for (6), for any $t < T$.

On the other hand, we are going to show that equation (3) admits a strong solution u at any time $t < T$. In particular, from the boundness of $\partial_t u$ it will follow that $g(t)$ is a Kähler metric for any $t < T$.

We will need the following

Lemma 3.4. *Let ψ be a non-negative $(1,1)$-form in an ample class on M. Given any $(1,1)$-form η, there exists η' (resp. η'') in the same class as η and a constant C such that*

$$\operatorname{tr}_\psi \eta' > C \quad (\textit{resp.}\ \operatorname{tr}_\psi \eta'' < C)$$

where it is defined.

Proof. Since ψ belongs to an ample class, there exists a positive $(1,1)$-form ψ_0 cohomologous to ψ, i.e. $\psi_0 = \psi + \mathrm{dd^c}\,\phi$ for some $\phi \in C^\infty(M)$.

Thus, we have

$$\begin{aligned}\operatorname{tr}_\psi \eta &= \operatorname{tr}\,(\psi^{-1} - \psi_0^{-1})\cdot\eta + \operatorname{tr}_{\psi_0}\eta \\ &= \operatorname{tr}\,(\psi^{-1}\cdot \mathrm{dd^c}\,\phi\cdot\psi_0^{-1}\cdot\eta) + \operatorname{tr}_{\psi_0}\eta.\end{aligned}$$

Given a constant a, we can define $\eta' = \eta + a\,\mathrm{dd^c}\,\phi$, that is a $(1,1)-$form cohomologous to η, such that

$$\begin{aligned}\operatorname{tr}_\psi \eta' &= \operatorname{tr}_\psi \eta + a\operatorname{tr}_\psi \mathrm{dd^c}\,\phi \\ &= \operatorname{tr}\,(\psi^{-1}\cdot\mathrm{dd^c}\,\phi)\cdot(\psi_0^{-1}\cdot\eta + a\,\mathrm{I}) + \operatorname{tr}_{\psi_0}\eta.\end{aligned}$$

By the positivity of ψ_0, it follows that in the locus and in the directions where ψ is zero, we have that $\mathrm{dd^c}\,\phi$ is positive. Therefore, if a is large enough so that $\psi_0^{-1}\eta + a\,\mathrm{I} > 0$, by the compactness of M, there exists a constant C such that $\operatorname{tr}_\psi \eta' > C$.

Similarly, by choosing $\eta'' = \eta - b\,\mathrm{dd^c}\,\phi$ with b large enough, there exists a constant C such that $\operatorname{tr}_\psi \eta'' < C$. □

Lemma 3.5. *By rescaling the initial metric g_0, by a positive constant K, the singular locus for the solution $g(t)$ of the Kähler-Ricci flow (1) at maximal time T, does not change.*

Proof. Let $g(t)$ be a solution for the Kähler-Ricci flow with initial metric g_0, and let $K > 0$.

If $\tilde g(s) = k(s)g(t(s))$, with

$$k(s) = (K-1)e^{-s} + 1 \qquad \text{and} \qquad t(s) = \log\left(\frac{e^s + K - 1}{K}\right),$$

then $\tilde g(s)$ is a solution for the rescaled Kähler-Ricci flow

$$\begin{cases} \tilde g'(s) = -\operatorname{Ric}_{\tilde g(s)} - \tilde g(s) \\ \tilde g(0) = K g_0 \end{cases} \tag{7}$$

In particular, we have that the singular locus of $g(\log(r+1))$ coincides with the singular locus of $\tilde g(\log(Kr+1))$. □

We can now prove:

Proposition 3.6. *For any $t_0 \in (0,T)$ there exist constants C_0, C, with C independent of t_0, such that, for any $t < t_0$ and as long as there exists a solution for (3), we have*

$$C_0 < \partial_t u < C$$

Proof. For ease of notation, let us denote $v = \partial_t u$. By taking the derivative of (3) with respect to t, we have that v is a solution of

$$\begin{cases} \partial_t v = \Delta_{g(t)} v + a'(t)\operatorname{tr}_{g(t)}\eta - v \\ v(0) = f \end{cases} \tag{8}$$

where $\Delta_{g(t)}$ denotes the Laplacian with respect to the metric $g(t)$.

Let us first show that there is a uniform upper bound for v. By rescaling A if necessary, by lemma 3.5 we can suppose without loss of generality that $A - K_M$ is ample. Thus we can choose η as a negative $(1,1)$-form in the class of $K_M - A$ and, since v is a solution of (8), it follows that for this choice of η, we have

$$\partial_t v \geq \Delta_{g(t)} v - v.$$

Thus by the Maximum Principle, v is uniformly bounded from above.

Let us suppose now that there exists $t_0 \in (0,T)$, such that, for any $t < t_0$, there exists a solution for (3) , but $\inf_M v(\cdot, t_0) = -\infty$.

Since $t_0 < T$, the non-negative $(1,1)-$form $\psi = g(t_0)$ belongs to the ample class $A(t_0)$. Therefore, by lemma 3.4, there exists a $(1,1)-$form η' in the class of $K_M - A$, and a constant C such that $\operatorname{tr}_\psi \eta' > C$. Moreover, since $g(t)$ is positive for any $t < t_0$, we can choose C so that $\operatorname{tr}_{g(t)} \eta' > C$ for any $t \leq t_0$.

We can suppose, without loss of generality, that $\eta = \eta'$. In fact, by lemma 2.3 the solution v' for (8), associated to η' will differ from v by $v' = v - a'(t)h$, for some C^∞ function h on M, not depending on t. Thus, the locus where the limit of $\inf_M v$ and of $\inf_M v'$ are $-\infty$ as $t \to t_0$, coincide.

Thus we have

$$\partial_t v \geq \Delta_{g(t)} v - v + a'(t)C$$

and by the Minimum Principle, it follows

$$\inf_M v(\cdot, t_0) \geq e^{-t_0}(Ct_0 + \min_M f) > -\infty,$$

which gives a contradiction, so v must be bounded from below on $M \times [0,t]$ for any $t < T$. □

We can now proceed to the C^1 and C^2 estimates. Let us denote by $\Delta = \Delta_{g_0(t)}$, the Laplacian with respect to the metric $g_0(t)$. Moreover let us denote by $\mathrm{R}_{i\bar{i}j\bar{j}}(g_0(t))$ the bisectional curvature of $g_0(t)$ (as mentioned above, we can choose η as in (5) so that $g_0(t) = g_0 + a(t)\eta$ is a metric for any $t < T$).

Lemma 3.7. *Let $t_0 \in (0,T)$. There exists a constant C_0, such that for any $t < t_0$ and as long as there exists a solution for (3), we have*

$$|\Delta u| \leq C_0.$$

Proof. Since $g(t) = g_0(t) + \mathrm{dd^c}\, u$, as long as there exists a solution $g(t)$, we must have

$$0 < tr_{g_0(t)} g(t) = n + \Delta u$$

and therefore the lower bound follows immediately from the fact $g(t)$ is a metric.

Let

$$F := \partial_t u + u - f - \log(\det(g_0(t)/g_0)). \tag{9}$$

Then

$$\det\left(g_0(t) + \mathrm{dd^c}\, u\right) = e^F \det(g_0(t)).$$

From (2.22) in [Yau], the following inequality holds for any $\lambda \in \mathbb{R}$:

$$\Delta_{g(t)}\left(e^{-\lambda u}(n + \Delta u)\right) \geq e^{-\lambda u}\big(\Delta F - n^2 \inf \mathrm{R}_{i\bar{i}j\bar{j}}(g_0(t)) \\ + e^{-\frac{F}{n-1}}(\inf \mathrm{R}_{i\bar{i}j\bar{j}}(g_0(t)) + \lambda)(n + \Delta u)^{\frac{n}{n-1}} - \lambda n(n + \Delta u)\big).$$

From now on, we choose λ to be any positive real number such that

$$\inf \mathrm{R}_{i\bar{i}j\bar{j}}(g_0(t)) + \lambda > 1.$$

Since $t_0 < T$, we can choose λ independent of t.

We have

$$\partial_t \Delta(\cdot) = \mathrm{tr}\left(g_0^{-1}(t)\ g_0'(t)\ g_0^{-1}(t)\ \mathrm{dd^c}(\cdot)\right).$$

Moreover, from (5) it follows that $rg_0'(T) \leq g_0(T)$, where r is given by (4) and $T = \log(r+1)$. Thus, there exists a constant C_0, such that $g_0'(t) \leq C_0\ g_0(t)$ for any $t \in (0, T)$. In particular, since $g_0(t) + \mathrm{dd^c}\, u > 0$, we have

$$\begin{aligned}(\partial_t \Delta)u &= \mathrm{tr}\left(g_0^{-1}(t)\ g_0'(t)\ g_0^{-1}(t)\ (g_0(t) + \mathrm{dd^c}\, u)\right) - tr_{g_0(t)} g_0'(t)\\ &\leq C_0 n(n + \Delta u) - \mathrm{tr}_{g_0(t)}\, g_0'(t).\end{aligned}$$

Thus, if $z = e^{-\lambda u}(n + \Delta u)$, we get

$$\begin{aligned}\partial_t z - \Delta_{g(t)} z\ \ &\leq e^{-\lambda u}(n + \Delta f + n^2 \inf \mathrm{R}_{i\bar{i}j\bar{j}}(g_0(t)) + \Delta \log\det(g_0(t)/g_0) - \mathrm{tr}_{g_0(t)}\, g_0'(t))\\ &\quad + (C_0 n - 1 + \lambda(n - \partial_t u))z - e^{\frac{\lambda u - F}{n+1}}(\inf \mathrm{R}_{i\bar{i}j\bar{j}}(g_0(t)) + \lambda)\ z^{\frac{n}{n-1}}\end{aligned} \tag{10}$$

By the Maximum Principle and since, by prop. 3.1 and 3.6, u and u_t are bounded function in $(0, t_0)$, the lemma follows. $\square$

From the above results, and from the Schauder estimate, we obtain a C^1-bound for the solution u. In fact, there exists a constant C such that

$$\sup_{M\times[0,T)} |\ \nabla u\ | \leq\ C\ (\sup_{M\times[0,T)} |\ \Delta u\ | + \sup_{M\times[0,T)} |\ u\ |)\ .$$

From the lemma and the fact that $g(t) = g_0(t) + \mathrm{dd^c}\, u$ is a metric, we also obtain a C^2-bound for u.

Moreover, from prop. 3.6 and from (3), we get

Lemma 3.8. *For any $t_0 \in (0, T)$ there exist positive constants C_0, C_1, such that, for any $t < t_0$ and as long as there exists a solution for (3), we have:*

$$C_0\ g_0 \leq g(t) \leq C_1\ g_0.$$

Now, similarly to [Cao], we can prove:

Theorem 3.9. *Let $T = \log(r+1)$, with r as in (4).*

There exists a bounded solution $u(x,t) \in C^\infty(M)$ for (3), at any time $t \in [0, T)$. Furthermore, since $u(x,t)$ are uniformly continuous in $[0,t)$, if $\lim_{t\to T} u(x,t)$ exists, it must be continuous.

4. Singularitites of the Limiting Metric

Let M be a projective manifold with canonical class K_M not nef, and let g_0 be a Kähler metric on M belonging to the ample class A. In the previous section we have shown that as long as $A(t)$ is an ample class, i.e. as long as $t < T = \log(r+1)$ with r as in (4), there exists a solution $g(t)$ for the Kähler-Ricci flow (1), that is in the class of $A(t)$.

On the other hand, at the finite time $T = \log(r+1)$, we have $A(T) = \frac{1}{r+1}L$, which is a non-positive, but semi-ample, line bundle. Thus, the associated pseudo-metric $g(T)$ will admit singularities, i.e. the set

$$S = \{x \in M | \nexists\, C_1, C_2 > 0 \text{ s.t. } \; C_1 g_0 \leq g(T) \leq C_2 g_0 \text{ locally at } x\}$$

is not empty.

From now on, we will suppose that M is of general type and therefore L is not only semi-ample, but also *big*, i.e., it defines a birational morphism

$$c : M \to M'$$

onto a projective variety M'. The exceptional locus E of c, i.e. the subset of M contracted by c, is the union of all the curves C, such that $L \cdot C = 0$. Thus E is the locus spanned by the extremal ray associated to L.

In general, we have

Lemma 4.1. *The singular locus S contains the exceptional set E. Moreover $g(T)$ is the pull-back of a (non-necessarily smooth) pseudo-metric on M' of the form $\alpha + dd^c u'$, where α is a smooth $(1,1)$-form and u' is continuous (hence L^1_{loc}).*

Proof. Since $g(T)$ belongs to the class of $\frac{1}{r+1}L$, it follows that for any curve C in M,

$$\int_C g(T) = \frac{L \cdot C}{r+1}.$$

$g(T)$, being a limit of positive $(1,1)$-forms, it is a non-negative $(1,1)-$form and therefore it follows that, for any curve C, such that $L \cdot C = 0$, we have that $g(T)$ is zero along the directions tangential to C. In particular, C is contained in S.

Since L is semi-ample, there exists an ample class A' on M' such that $L = c^* A'$. Thus if we choose η_L to be the pull back of a non-negative $(1,1)$-form in A', as in section 3, we can write $g(t) = g_0(t) + \mathrm{dd^c}\, u$, with

$$g_0(t) = \frac{1}{r}(a(t)\eta_L + b(t) g_0).$$

Since both $g_0(T) = \frac{1}{r+1}\eta_L$ and $g(T)$ are zero along the direction tangential to C, it follows that $\mathrm{dd^c}\, u = 0$ along C and therefore $\Delta_C u = 0$ (here Δ_C is the complex Laplacian on C). Thus, since C is compact, u is smooth along C (by *elliptic regularity*) and it induces a function u' on M'. Since u is continuous, it must be the case that also u' is. Thus $g(T) = \eta_L + c^*(\mathrm{dd^c}\, u')$ is the pull-back of a $(1,1)$ form on M' with the required properties. □

Remark 4.2. Note that this lemma holds also in the case that L is only semi-ample (i.e., without requiring that K_M be big), in that case though, one would get a map $c : M \to S$ with $\dim(S) < \dim(M)$ and $g(T)$ would be degenerate along the fibers.

We will need the following:

Lemma 4.3. *Let $f : M \to N$ be a smooth holomorphic map of Kähler manifolds, and let $g = g_N$ be a Kähler metric on N such that $h = f^* g_N \geq 0$. Then $\mathrm{R}_{i\bar{i}j\bar{j}}(h)$ is bounded.*

Proof. The boundedness of the bisectional curvature being a local matter, we can pick a point $p \in M$ and show the boundedness around that point.

Since the property of being bounded is independent of the coordinates chosen, we can choose coordinates $\{z_1, \cdots, z_n\}$ centered around p and coordinates $\{w_1, \cdots, w_n\}$ centered around $f(p)$ such that $h = \sum_i \lambda_i dz_i \wedge dz_{\bar{i}}$ and $g = \sum_i \mu_i dw_i \wedge dw_{\bar{i}}$. In particular:

$$\lambda_k = \sum_{i=1}^{n} \mu_i \frac{\partial f_i}{\partial z_k} \overline{\frac{\partial f_i}{\partial z_k}}$$

and since g is positive, this vanishes exactly if and only if $|\frac{\partial f_i}{\partial z_k}|^2 = \frac{\partial f_i}{\partial z_k}\overline{\frac{\partial f_i}{\partial z_k}} = 0$ for every i. Let i_o be such that $|\frac{\partial f_{i_o}}{\partial z_k}|^2$ has minimal order of vanishing at p (this is the order of vanishing of λ_k)

One computes:

$$\frac{\partial \lambda_k}{\partial z_j} = \sum_{i=1}^{n} \frac{\partial \mu_k}{\partial z_j} \frac{\partial f_i}{\partial z_k} \overline{\frac{\partial f_i}{\partial z_k}} + \sum_{i=1}^{n} \mu_k \frac{\partial^2 f_i}{\partial z_j \partial z_k} \overline{\frac{\partial f_i}{\partial z_k}}$$

and the analogous formula for $\frac{\partial \lambda_k}{\partial \bar{z}_j}$:

$$\frac{\partial \lambda_k}{\partial \bar{z}_j} = \sum_{i=1}^{n} \frac{\partial \mu_k}{\partial \bar{z}_j} \frac{\partial f_i}{\partial z_k} \overline{\frac{\partial f_i}{\partial z_k}} + \sum_{i=1}^{n} \mu_k \frac{\partial f_i}{\partial z_k} \overline{\frac{\partial^2 f_i}{\partial z_h \partial z_k}}.$$

Therefore, $\frac{\partial \lambda_k}{\partial z_j} = |\frac{\partial f_{i_o}}{\partial z_k}|^2 A_1 + \frac{\partial}{\partial z_j}\left(|\frac{\partial f_{i_o}}{\partial z_k}|^2\right) B_1$ and $\frac{\partial \lambda_k}{\partial \bar{z}_j} = |\frac{\partial f_{i_o}}{\partial z_k}|^2 A_2 + \frac{\partial}{\partial \bar{z}_j}\left(|\frac{\partial f_{i_o}}{\partial z_k}|^2\right) B_2$

From this it is easy to see that $\frac{\partial \lambda_k}{\partial z_j}\frac{\partial \lambda_k}{\partial \bar{z}_j}$ is divisible by $|\frac{\partial f_{i_o}}{\partial z_k}|^2$.

Since in these coordinates the bisectional curvature equals:

$$\mathrm{R}_{k\bar{k}j\bar{j}}(h) = -\frac{\partial^2 \lambda_k}{\partial z_j \partial \bar{z}_j} + \frac{1}{\lambda_k} \sum_k \frac{\partial \lambda_k}{\partial z_j} \frac{\partial \lambda_k}{\partial \bar{z}_j},$$

we confirm the conclusion of the theorem. □

In order to have a better understanding of the singularities of the pseudo-metric induced on M', we will first study the locus:

$$S_0 = \{x \in M |\det(g(T)) = 0\}.$$

Obviously S_0 is contained in the singular locus S for $g(T)$.

Although not quite standard, we will call a birational contraction $c : M \to M'$ *divisorial* if the exceptional locus E of c is of pure codimension 1 and M' is $\mathbb{Q}-$factorial, i.e. for any Weil divisor D on M', there exists an integer m such that mD can be locally defined by one equation. This last requirement is superfluous if c is an extremal contraction. Moreover in the case of a projective surface S, the contraction of any K_S-negative curve is divisorial.

Proposition 4.4. *The locus S_0 where $\det g(T) = 0$ is contained in a proper sub-variety of M. Moreover, if $c : M \to M'$ is a divisorial contraction, then the locus S_0 coincides with the exceptional locus E (when viewed as sets).*

Proof. We will first show that $S_0 \subset F$ for some analytic subvariety F. Let $A = A(0)$ be the class of the initial metric g_0. Since K_M is big, by Riemann-Roch it follows that there exists a positive integer m and a $\mathbb{Q}-$effective $\mathbb{Q}-$divisor D (i.e. dD is an effective integral divisor for some positive integer d) such that $mK_M = A + D$.

By lemma 3.5, we can rescale g_0 by $1/m$, so that we can suppose without loss of generality, that $m = 1$ and $K_M = A + D$.

Let us choose $\eta_0 = -g_0 - \mathrm{Ric}_{g_0}$. For this choice, in the class of $K_M - A$, the equation (3) becomes:

$$\partial_t u = \log\ \det\left(\frac{g_0 + a(t)\eta_0 + \mathrm{dd^c}\, u}{g_0}\right) - u \tag{11}$$

Therefore, by differentiating with respect to t, we have, as in (8), that $v = \partial_t u$ is a solution of:

$$\begin{cases} \partial_t v = \Delta_{g(t)} v + a'(t) \operatorname{tr}_{g(t)} \eta_0 - v \\ v(x,0) = 0 \end{cases} \tag{12}$$

From (11) and since by proposition 3.1 the function u is uniformly bounded in $(0,T)$, it follows immediately that the locus S_0 where $\det g(T) = 0$, coincides with the locus where $\lim_{t\to T} v = -\infty$.

Since η_0 is in the class of $c_1(D)$, and D is $\mathbb{Q}-$effective, there exists a positive integer d, such that $d\eta_0$ is in the class of the effective divisor dD.

Let σ be a non zero holomorphic section in $\mathrm{H}^0(M, dD)$ with zero locus F, and let h be any hermitian metric on dD. Then, by Chern theory, there exists a smooth function ϕ_0 on M such that $\frac{\sqrt{-1}}{2\pi}\Theta_h(D) = \eta_0 + \mathrm{dd^c}\,\phi_0$.

Moreover, by the Lelong-Poincaré formula (e.g. see (3.11) in [Demailly]) we have

$$\mathrm{dd^c} \log(\|\sigma\|_h^2) = [F] - \frac{\sqrt{-1}}{2\pi}\Theta_h(dD)$$

where the equality is understood in the sense of currents and $[F]$ denotes the current of integration associated to F, defined by, for any $(n-1,n-1)-$form α,

$$< [F], \alpha >= \int_F \alpha.$$

Therefore if $\phi = \phi_0 + \frac{1}{d}\log(\|\sigma\|_h^2)$ on $M \setminus F$, we have

$$\eta_0 = \frac{1}{d}[F] - \mathrm{dd^c}\,\phi \tag{13}$$

Let $\overline{v} = v - a'(t)\phi$, then $\overline{v}(\cdot,t)$ is a smooth function on $M \setminus F$, defined for any $t < T$. Moreover, since $a''(t) = -a'(t)$, we have that $\overline{v}$ satisfies the equation:

$$\begin{aligned} \partial_t \overline{v} &= \partial_t v - a''(t)\phi \\ &= \Delta_{g(t)} v + a'(t) \operatorname{tr}_{g(t)} \eta_0 - v + a'(t)\phi \\ &= \Delta_{g(t)} \overline{v} + a'(t) \operatorname{tr}_{g(t)} \mathrm{dd^c}\,\phi + a'(t) \operatorname{tr}_{g(t)} \eta_0 - \overline{v} \end{aligned}$$

Thus by (13), it follows that $\overline{v}$ is a solution of

$$\begin{cases} \overline{v}_t = \Delta_{g(t)} \overline{v} - \overline{v} & \text{on } (M\setminus F) \times (0,T) \\ \overline{v} = -\phi & \text{on } (M \setminus F) \times \{t = 0\} \end{cases} \tag{14}$$

Since for any $t \in [0,T)$ and for any x approaching F, we have that $\overline{v}(x,t) \to +\infty$, then the function $\overline{v}$ must admit a minimum inside $M \setminus F$ for any time t. Therefore by the Minimum Principle, applied at (14), it follows that

$$\overline{v}(t) \geq \inf \overline{v}(\cdot, 0) = -\max\phi > -\infty.$$

Therefore $v = \overline{v} + a'(t)\phi \geq -\max\phi + a'(t)\phi$, and in particular the locus S_0 where $v \to -\infty$ must be contained inside F (since $a'(t) = e_t$ is bounded in t, for $t \geq 0$).

Let us now prove that under the assumption that $c : M \to M'$ is a divisorial contraction, then the locus S_o coincides with the exceptional locus E of c. Let E_i be the irreducible components of E. Then, since we are assuming that M' is $\mathbb{Q}-$factorial, the relative Picard group of c is generated by E_i, i.e.

$$\mathrm{Pic}(M)/c^*\,\mathrm{Pic}(M') =< E_i > .$$

Therefore $A = c^*B - \sum \delta_i E_i$, for some ample class B on M' and positive constants δ_i.

If α_i is the discrepancy of E_i with respect to the morphism c, i.e. if

$$\alpha_i = \mathrm{mult}_{E_i}(K_{M/M'}),$$

then, since M' has terminal singularities (e.g. see [Kawamata-Matsuda-Matsuki]), it follows that $\alpha_i > 0$.

Thus

$$L = rK_M + A = c^*(rK_{M'} + B) + \sum (r\alpha_i - \delta_i)E_i,$$

from which it follows that $r\alpha_i - \delta_i = 0$.

Therefore one has:

$$K_M - A = c^*(K_{M'} - B) + (r+1)\sum \alpha_i E_i.$$

From lemma 4.1, it follows that $g(T)$ is the pull-back of a non-negative $(1,1)$-form g' in an ample class of M', therefore, it follows from lemma 3.4 that we can choose a $(1,1)$-form η' in the class of $K_{M'} - B$, such that

$$\mathrm{tr}_{g'}\, \eta' > C, \tag{15}$$

for some constant C.

Thus, similarly to (13), there exists a smooth function ϕ_0, such that if

$$\phi = \phi_0 + (r+1)\sum \alpha_i \log(\|\sigma_i\|^2_{h_i}),$$

where $\sigma_i \in \mathrm{H}^0(M, E_i)$ are non-zero holomorphic sections and h_i are hermitian metrics on E_i, then

$$\eta_0 = (r+1)\sum \alpha_i [E_i] + c^*\eta' - \mathrm{dd^c}\, \phi.$$

Thus, from (15), equation (14) becomes

$$\begin{cases} \overline{v}_t \geq \Delta_{g(t)}\overline{v} - \overline{v} + C & \text{on } (M\setminus E)\times(0,T) \\ \overline{v} = -\phi & \text{on } (M\setminus E)\times\{t=0\} \end{cases}$$

and similarly to what we did earlier, it follows that there exists a lower bound for $\overline{v}$. Thus, since $v = \overline{v} + a'(t)\phi$, and $T = \log(r+1)$, there exists a constant B such that

$$\begin{aligned} \partial_t u_{|t=T} = v_{|t=T} &\geq B + a'(T)(r+1)\sum \alpha_i \log(\|\sigma_i\|^2_{h_i}) = \\ &= B + \sum \alpha_i \log(\|\sigma_i\|^2_{h_i}) \end{aligned}$$

from which it follows that v is bounded outside E.

Therefore, since by proposition 3.1 the function u is bounded, eq. (11) yields the existence of a positive constant B' such that

$$\det(g(T)/g_0) \geq B' \prod \|\sigma_i\|^{2\alpha_i}_{h_i}. \tag{16}$$

In particular, the locus S_0 where $\det g(T) = 0$ coincides with E. □

We are now ready to prove the main result of this section

Theorem 4.5. *If $c : M \to M'$ is a divisorial contraction and M' is smooth, then the singular locus S for $g(T)$ coincides with the exceptional locus E. Moreover $g(T)$ induces a smooth metric on M'.*

We will need the following:

Lemma 4.6. *If the bisectional curvature* $\mathrm{R}_{i\bar{i}j\bar{j}}$ *of* $g_0(t)$ *is uniformly bounded in* $[0,T)$*, then there exists a constant* C *such that*

$$\Delta_{g_0(t)}u + \log(\det(g_0(t)/g_0)) < C.$$

Proof. Let

$$\overline{z} = e^{-\lambda u}(n + \Delta_{g_0(t)}u + \log\det(g_0(t)/g_0) + f).$$

where, as in the proof of lemma 3.7, λ is a postitive constant such that

$$\inf \mathrm{R}_{i\bar{i}j\bar{j}}(g_0(t)) + \lambda > 1.$$

Then, by equation (10), there exist constant C_0, C_1, C_2 with $C_2 < 0$ and such that

$$\partial_t\overline{z} - \Delta_{g(t)}\overline{z} \ \le e^{-\lambda u}(n + n^2 \inf \mathrm{R}_{i\bar{i}j\bar{j}}(g_0(t)) + C_0) + C_1\overline{z} + C_2\overline{z}^{\frac{n}{n-1}}$$

Thus, the claim follows immediately by the Maximum Principle. □

Proof of Theorem 4.5.

Let η_L be induced by a smooth metric on M'. From lemma 4.3, it follows that the bisectional curvature of $g_0(t)$ is uniformly bounded. Thus by lemma 4.6, the metric $g(t)$ remains bounded outside the exceptional locus of $c : M \to M'$.

Moreover, there exists a smooth function ψ on M, such that, using the same notation as in the proof of proposition 4.4 , we have

$$\det(\eta_L/g_0) = e^{\psi} \prod \|\sigma_i\|_{h_i}^{2\alpha_i}. \tag{17}$$

In particular by (16), it follows that there exists a positive constant B_0 such that

$$\det(g(t)/g_0(t)) > B_0,$$

for any $t \in (0,T)$. We now want to show that there exists an upper bound for $\det(g(t)/g_0(t))$. Let $\phi = e^t \log\det(g_0(t)/g_0)$ and $w = e^t\partial_t u - \phi$. Then, equation (8) becomes

$$\begin{cases} \partial_t w = \Delta_{g(t)}w + \mathrm{tr}_{g(t)}(\eta + \mathrm{dd^c}\,\phi) - \phi' \\ w(x,0) = f \end{cases} \tag{18}$$

where η depends on η_L, as in (5). Let us study the term $\Phi = \mathrm{tr}_{g(t)}(\eta + \mathrm{dd^c}\,\phi) - \phi'$ in equation (18). We have:

$$\Phi = \mathrm{tr}[(g^{-1}(t) - g_0^{-1}(t))\eta] + \mathrm{tr}_{g(t)}\,\mathrm{dd^c}\,\phi - \phi.$$

From (17) we have

$$\mathrm{dd^c}\,\phi_{|t=T} = (r+1)\sum \alpha_i\,\mathrm{dd^c}\log||\sigma_i||_{h_i}^2$$

in fact, we have that there exists a $\delta > 0$ and constants A and B such that for every $t \in (T-\delta, T)$, one has:

$$\mathrm{dd^c}\,\phi \le Ae^t \sum \alpha_i\,\mathrm{dd^c}\log||\sigma_i||_{h_i}^2$$

and

$$\phi \le Be^t \sum \alpha_i\,\mathrm{dd^c}\,||\sigma_i||_{h_i}^2$$

from which it easely follows that every term of ψ is integrable with respect to t in $(0,T)$, and by the Maximum Principle it follows that the solution w for (18) is bounded. In particular $\det(g(t)/g_0(t))$ is uniformly bounded. □

5. Maximal existence time. Case II: K_M nef

In this section, we consider the case of a smooth projective variety M such that the canonical class K_M is nef. As above, let A be the ample class that represents the initial metric g_0.

By prop. 2.1 the solution $g(t)$ for the Kähler-Ricci flow (1) belongs to the class

$$A(t) = A + a(t)(K_M - A),$$

where $a(t) = 1 - e^{-t}$. Since K_M is nef, it follows that $A(t)$ is ample for any $t > 0$.

Moreover, by the base point free theorem, we have that $L = K_M$ is semi-ample and therefore, as in 5, we can write

$$g_0 = a(t)\eta_L + b(t)g_0, \tag{19}$$

where η_L is a non-negative $(1,1)$-form in the class of K_M and $b(t) = e^{-t}$.

Thus the solution $g(t)$ can be written as $g(t) = g_0(t) + \mathrm{dd^c}\, u$, for some function u that satisfies equation (3).

Proposition 5.1. *If M is a projective variety with nef canonical line bundle, then the Kähler-Ricci flow (1) admits a smooth solution $g(t)$ for any time $t \in (0, +\infty)$.*

Proof. As in lemma 3.2, we are going to show the existence of a bounded super-solution u^+ and a bounded sub-solution u^- for (3).

Since $a(t)$ and $b(t)$ are bounded and M is compact, there exists a constant K, such that $\log\det(g_0(t)) + f < K$. Therefore, as in lemma 3.2, in order to find a super-solution u^+, it is enough to solve $\partial_t u^+ + u^+ = K$, with $u^+(0) = 0$.

Similarly, for any $t_0 < +\infty$, it is easy to find a bounded and space-independent sub-solution u^- (depending on t_0), defined for any $t \in (0, t_0)$.

Moreover, as in proposition 3.6, $v = \partial_t u$ is a solution for

$$\begin{cases} \partial_t v = \Delta_{g(t)} v + a'(t)\,\mathrm{tr}_{g(t)}(\eta_L - g_0) - v \\ v(0) = f \end{cases} \tag{20}$$

By lemma 3.4, for any $t_0 < +\infty$, we can modify the equation (20), by choosing a $(1,1)-$form η' (resp. η'') cohomologous to $\eta_L - g_0$ and such that $\mathrm{tr}_{g(t)}\,\eta' > C$ (resp. $\mathrm{tr}_{g(t)} < C$) for some constant C. From that, it follows that v is bounded in the interval $(0, t_0)$.

Following exactly the same lines as in section 3, we obtain the same C^1 and C^2 estimates for u. Thus, for any $t_0 \in (0, +\infty)$, there exists a bounded solution u for (3) in $(0, t_0)$.

Thus, the claim follows. □

From the proof of the previous proposition, it follows that the solution u for (3) is uniformly bounded from above for any $t \in (0, +\infty)$.

As in the previous section, we will suppose from now on, that K_M is also big. Since by the nefness assumption, K_M is semi-ample, it defines a birational morphism $c : M \to X$, onto the canonical model X for M. By a classic result in algebraic geometry, X has canonical singularities, and K_X is ample (e.g. see [Kawamata-Matsuda-Matsuki]).

In order to study the singularities for the limit metric for the Kähler-Ricci flow in this situation, we are going to restrict our self to the case X admits only orbifold singularities. In particular, this always holds if M is a surface of general type.

Thus, we have

Proposition 5.2. *Let M be a projective manifold of general type with nef canonical bundle and such that its canonical model admits only orbifold singularitites.*

Then equation (3) admits a uniformly bounded solution for any $t \in (0, +\infty)$.

Proof. By the proof of proposition 5.1, and by the comparison principle, it is enough to show that there exists a bounded sub-solution in some interval $(t_0, +\infty)$.

Since K_M is semi-ample, there exists a smooth non-negative $(1,1)$-form η_L in the class of K_M. Moreover, since by assumption X is an orbifold, we can choose $\tilde{\eta}_L$ as the pull-back of an orbifold metric on X (e.g. see [Demailly-Kollár]). In particular, if $c : M \to X$ is the canonical map defined by K_M, since $K_M = c^* K_X$, there exists a smooth function ψ, such that

$$\det \tilde{\eta}_L = \det g_0 \cdot e^{-\psi}. \tag{21}$$

Moreover, there exists a continuous function ϕ_0 (that is smooth in the orbifold sense), such that

$$\tilde{\eta}_L = \eta_L + \mathrm{dd^c}\, \phi_0.$$

It is possible to approximate ϕ_0 by a sequence of smooth (in the ordinary sense) function on M (e.g. see [Baily]). Therefore we can choose $\phi(t) \in C^\infty(M)$ uniformly bounded and such that

$$\lim_{t\to\infty} \phi(t) = \phi_0 \qquad \text{and} \qquad \lim_{t\to\infty} \mathrm{dd^c}\, \phi(t) = \mathrm{dd^c}\, \phi_0.$$

For any $\rho \in (0,1)$, let us define

$$G_\rho := \log\det \rho \cdot \left(\frac{g_0(t) + \mathrm{dd^c}\, \phi(t)}{g_0} \right) + \psi$$

with $g_0(t)$ as in (19). Then, by (21), it follows that $\lim_{t\to\infty} G_\rho = n \log \rho$ is a constant.

Fixed a constant

$$C \geq \sup_{M\times(0,+\infty)} (\phi(t) + \phi'(t) - f),$$

let

$$u^- := e^{-t} \cdot \int_0^t e^s G_\rho \; ds + \phi(t) - C.$$

Clearly u^- is bounded. We want to show that if t_0 is sufficiently large (depending on ρ), then u^- is a sub-solution for (3) in the interval $(t_0, +\infty)$. This will follow from:

Claim: There exists t_0 such that

$$\det(\rho(g_0(t) + \mathrm{dd^c}\, \phi(t))) \leq \det(g_0(t) + \mathrm{dd^c}\, u^-) \qquad \text{for } t \in (t_0, +\infty) \tag{22}$$

In fact, we have

$$\begin{aligned} \partial_t u^- + u^- &= G_\rho + \phi'(t) + \phi(t) - C \\ &\leq \log\det \rho \left(\frac{g_0(t) + \mathrm{dd^c}\, \phi(t)}{g_0} \right) + f \\ &\leq \log\det \left(\frac{g_0(t) + \mathrm{dd^c}\, u^-}{g_0} \right) + f. \end{aligned}$$

where the second line follows from the choice of C, while the third line follows from (22). Thus u^- is a sub-solution for (3), and in particular there exists a constant C' such that $u \geq u^- + C'$, for any $t \in (t_0, +\infty)$.

In order to prove the claim, it is enough to observe that by taking the limit $t \to +\infty$, we have

$$\begin{aligned}\lim_{t\to+\infty} \det(\rho(g_0(t) + \mathrm{dd^c}\,\phi(t))) &= \det(\rho(\eta_L + \mathrm{dd^c}\,\phi_0)) \\ &< \det(\tilde{\eta}_L) \\ &= \lim_{t\to+\infty} \det(g_0(t) + \mathrm{dd^c}\,u^-).\end{aligned}$$

Thus, by the compactness of M, the claim follows. □

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